

Replacing the Breit-Wigner amplitude by the complex delta function to describe resonances

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Whenever the Breit-Wigner amplitude appears in a calculation, there are many instances (e.g., Fermi's two-level system and the Weisskopf-Wigner approximation) where energy integrations are extended from the scattering spectrum of the Hamiltonian to the whole real line. Such extensions are performed in order to obtain a desirable, causal result. In this paper, we recall several of those instances and show that substituting the Breit-Wigner amplitude by the complex delta function allows us to recover such desirable results without having to extend energy integrations outside of the scattering spectrum.

§1. Introduction

As is well known, the decay/resonant amplitude of a resonance cannot exactly coincide with the Breit-Wigner amplitude. One reason is that the Breit-Wigner amplitude yields the exponential decay law only when it is defined over the whole of the energy real line $(-\infty, \infty)$ rather than just over the scattering spectrum. Because in quantum mechanics the scattering spectrum has a lower bound, the Breit-Wigner amplitude would yield the exponential decay law only if it was defined also at energies that do not belong to the scattering spectrum.

Another reason why the Breit-Wigner amplitude cannot exactly coincide with the resonant amplitude is that the energy (i.e., spectral) representation of the Gamow state is given by the complex delta function rather than by the Breit-Wigner amplitude. Because the Gamow state is the natural wave function of a resonance, the exact resonant amplitude is given by the complex delta function.

Even though it is well known that it cannot exactly coincide with the resonant amplitude, the Breit-Wigner amplitude is often used to describe the decay of unstable systems. Two classic examples are Fermi's two-level system and the Weisskopf-Wigner approximation. However, whenever it is used in such examples, the Breit-Wigner amplitude is extended from the scattering spectrum to the whole real line of energies in order to obtain some desirable, causal results.

Because the exact resonant amplitude is given by the complex delta function, one may wonder if such desirable results can be recovered by way of the complex delta function without using the approximation of extending energy integrations to the whole real line. The purpose of this paper is to show how this can be done.

In Secs. 2, 3 and 4, we recall, respectively, the main features of Fermi's two-level system, a standard treatment of unstable states, and the Weisskopf-Wigner approximation. In Sec. 5, we explain why the complex delta function gives us the same results as the Breit-Wigner amplitude without extending energy integrations outside of the scattering spectrum. In Sec. 6, we state our conclusions.

§2. Fermi's two level system

In 1932, Fermi constructed a simple model to check whether Quantum Mechanics is compatible with Einstein causality.¹⁾ He considered a pair of two atoms A and B separated by a distance R . The states of each atom form a two-level system (see Fig. 1a). The energy gap of each two-level system is hu (see Fig. 1a). The initial state is such that atom A is in the excited state, whereas atom B is in the ground state (see Fig. 1a). When atom A decays to its ground state, it emits a photon of energy hu . This photon may eventually hit atom B, causing atom B to reach the excited state. The final state is such that atom A is in the ground state, whereas atom B is in the excited state (see Fig. 1b). Fermi then calculated the probability $\mathcal{P}_{i \rightarrow f}(t)$ of going from the initial state of Fig. 1a to the final state of Fig. 1b. According to Einstein causality, $\mathcal{P}_{i \rightarrow f}(t)$ should be zero for any instant t less than R/c , i.e., for any t less than what it takes the photon to go from atom A to atom B (see Fig. 2a). This is the result that Fermi obtained.¹⁾

About the same time Fermi proposed this model, von Neumann published his book on the mathematical foundations of Quantum Mechanics.²⁾ According to von Neumann, the energy observable is represented by a linear, self-adjoint operator, called Hamiltonian, that acts on a Hilbert space. The spectrum of the Hamiltonian, which is identified with the physical spectrum, should be bounded from below (i.e., semibounded).

In 1966, Shirokov³⁾ pointed out that, in order to obtain the result of Fig. 2a, Fermi had approximated an integral over positive energies (i.e., over the scattering spectrum) by an integral over the full energy real line $(-\infty, \infty)$. Such integral involves the Breit-Wigner amplitude. This approximation is *crucial* to Fermi's calculation: if the integral is performed over the scattering spectrum, then the causal result of Fig. 2a does not hold.³⁾ In fact, in 1994 Hegerfeldt⁴⁾ showed, in a model independent manner, that the problem pointed out by Shirokov within Fermi's system is quite general: the semiboundedness of the Hamiltonian leads to conflicts with causality. More precisely, according to Hegerfeldt's theorem, Quantum Mechanics predicts that either atom A never decays (see Fig. 2b), or else there is a non-zero probability that atom B reaches the excited state before the photon from atom A can possibly arrive at atom B (see Fig. 2c).

§3. Unstable states

Approximations similar to Fermi's approximation can be found in standard textbooks dealing with unstable states. For example, in Sec. 13.d of Ref.,⁵⁾ Taylor uses such kind of approximation when dealing with the decay of a resonant state. More precisely, Taylor's equation (13.3) reads as

$$\psi_{\text{sc}}(\mathbf{x}, t) = \text{constant } \Gamma Y_l^0(\hat{\mathbf{x}}) \phi_l(E_R) \frac{e^{i(p_R r - E_R t)}}{p_R^{1/2} r} \times \int_0^\infty dE \frac{e^{i(E - E_R)(t - r/v_R)}}{E - E_R + i\Gamma/2}, \quad (3.1)$$

where $z_R = E_R - i\Gamma/2$ is the complex resonant energy. Taylor then continues by saying that "the integral can be extended to $-\infty$ without significantly affecting its

value.” After such extension, Taylor obtains the following desirable result:

$$|\psi_{\text{sc}}(\mathbf{x}, t)|^2 = 2\pi m \Gamma^2 |Y_l^0(\hat{\mathbf{x}})|^2 |\phi_l(E_R)|^2 \frac{e^{-\Gamma(t-r/v_R)}}{p_R r^2} \theta\left(t - \frac{r}{v_R}\right). \quad (3.2)$$

Equation (3.2) implies that the decay of a resonant state follows the exponential decay law in a causal manner. Clearly, Eq. (3.2) does not hold exactly when the integration is done over the scattering spectrum, just like causality is not preserved in Fermi’s two-level system when the integration is done over the scattering spectrum.

In Ref.,⁽⁶⁾ Ballentine treats the decay of a resonance in a similar way to Taylor. Ballentine also extends an energy integral to the whole real line (see Eq. (16.120) of Ref.⁽⁶⁾) in order to obtain a desirable, causal result.

§4. Weisskopf-Wigner approximation

In quantum mechanics, the approximation of extending the range of the Breit-Wigner amplitude to the whole real line is often referred to as the Weisskopf-Wigner approximation. Such approximation is used in many calculations. For example, in Ref.,⁽⁷⁾ Scully and Zubairy calculate the following amplitude for the first-order correlation function:

$$\langle 0|E^{(+)}(\mathbf{r}, t)|\gamma_0\rangle = \frac{ic\mathcal{P}_{ab}\sin\eta}{8\pi^2\epsilon_0\Delta r} \times \int_0^\infty dk k^2 (e^{ik\Delta r} - e^{-ik\Delta r}) \frac{e^{-i\nu_k t}}{(\nu_k - \omega) + i\Gamma/2}. \quad (4.1)$$

Then, Scully and Zubairy extend the range of the integral to the whole real line and obtain a desirable causal result for the first-order correlation function:

$$G^{(1)}(\mathbf{r}, \mathbf{r}; t, t) = |\langle 0|E^{(+)}(\mathbf{r}, t)|\gamma_0\rangle|^2 = \frac{|\mathcal{E}_0|^2}{|\mathbf{r} - \mathbf{r}_0|^2} \theta\left(t - \frac{|\mathbf{r} - \mathbf{r}_0|}{c}\right) e^{-\Gamma(t-|\mathbf{r}-\mathbf{r}_0|/c)}. \quad (4.2)$$

As in the above examples, this result cannot be obtained unless the range of the frequency (energy) integration in Eq. (4.1) is extended to the whole real line.

§5. Substituting the Breit-Wigner amplitude by the complex delta function

From the above examples, we have seen that, whenever we describe the decay of an unstable state by the Breit-Wigner amplitude, we arrive at an integral of the form

$$\int_0^\infty dE e^{-iEt} \frac{f(E)}{E - z_R}, \quad (5.1)$$

where $f(E)$ is an analytic function of E , and $z_R = E_R - i\Gamma/2$ is the resonant energy. By assuming that the extension of the integral to the whole real line makes little error, one gets

$$\int_{-\infty}^\infty dE e^{-iEt} f(E) \frac{1}{E - z_R} = \frac{2\pi}{i} f(z_R) e^{-iE_R t} e^{-\Gamma t/2}, \quad t > 0. \quad (5.2)$$

Equations similar to (5.2) are widely used in the literature on resonances (see e.g. review⁸⁾).

Clearly, the desirable result (5.2) is obtained by using the approximation of extending the energy integration to the whole real line. It seems therefore pertinent to try to recover (5.2) as an exact result. In order to do so, we are going to substitute the Breit-Wigner amplitude by the complex delta function.

The complex delta function was introduced by Nakanishi⁹⁾ to describe resonances in the Lee model,¹⁰⁾ and it has been used by a number of authors (see e.g. Refs.^{11), 12), 13)} and references therein). As shown in Ref.,¹⁴⁾ describing resonances by means of the complex delta function is the same as describing resonances by means of the Gamow state.^{15), 16), 17), 18), 19), 20), 21), 22), 23), 24), 25), 26), 27), 28), 29), 30), 31), 32), 33), 34), 35), 36), 37), 38), 39), 40), 41),}

We recall that the Gamow states are eigenfunctions of the Hamiltonian subject to purely boundary conditions. The eigenvalue of the Gamow state is also a pole of the S matrix. The resonant amplitude associated with the Gamow states is given by the complex delta function, and the Breit-Wigner amplitude is just an approximate resonant amplitude that is valid whenever we neglect the lower bound of the energy.*)

Mathematically, the complex delta function is a distribution that associates, with a test function g , the value of such function at $z = z_R$:

$$\int_0^\infty dE g(E) \delta(E - z_R) = g(z_R). \quad (5.3)$$

Now, if the resonant amplitude is given by the complex delta function (rather than by the Breit-Wigner amplitude $\frac{1}{E - z_R}$), and if the scattering spectrum is the positive real line (rather than the whole real line), Eq. (5.1) should be written as

$$\int_0^\infty dE e^{-iEt} f(E) \delta(E - z_R). \quad (5.4)$$

By combining Eqs. (5.3) and (5.4), we obtain

$$\int_0^\infty dE e^{-iEt} f(E) \delta(E - z_R) = f(z_R) e^{-iE_R t} e^{-\Gamma t/2}, \quad (5.5)$$

which, up to a numerical factor, coincides with the desirable result (5.2). In addition, since the time evolution of the complex delta function is defined only for $t > 0$ (see Refs.^{11), 12), 13)}), Eq. (5.5) is valid only for $t > 0$:

$$\int_0^\infty dE e^{-iEt} f(E) \delta(E - z_R) = f(z_R) e^{-iE_R t} e^{-\Gamma t/2}, \quad t > 0. \quad (5.6)$$

Thus, instead of using the Breit-Wigner amplitude and integrating over the whole real line as in Eq. (5.2), we can integrate over the scattering spectrum and use the complex delta function as in Eq. (5.6) to obtain the same result.

*) One may argue that the Breit-Wigner amplitude is an approximation also because the exact formula corresponding to the denominator $E - z_R$ is a much more complicated function of E that includes a self-energy contribution.

§6. Conclusion

In Quantum Mechanics, the combination of two approximations –the approximation of describing the decay of an unstable state by means of the Breit-Wigner amplitude, and the approximation of extending the Breit-Wigner amplitude to the whole real line– yields desirable, causal results for the decay of a resonance. In this paper, we have seen that if we replace the Breit-Wigner amplitude by the complex delta function, it is possible to recover such desirable results without the need to extend any energy integration outside of the physical scattering spectrum. This result provides another argument in favor of seeing the complex delta function as the exact resonant amplitude, and the Breit-Wigner amplitude as an approximate resonant amplitude that is valid whenever we can neglect the lower bound of the energy.¹⁴⁾ This results also shows that what Fermi's, Weisskopf-Wigner's and similar approximations do is basically to neglect the effect of the lower bound of the energy.

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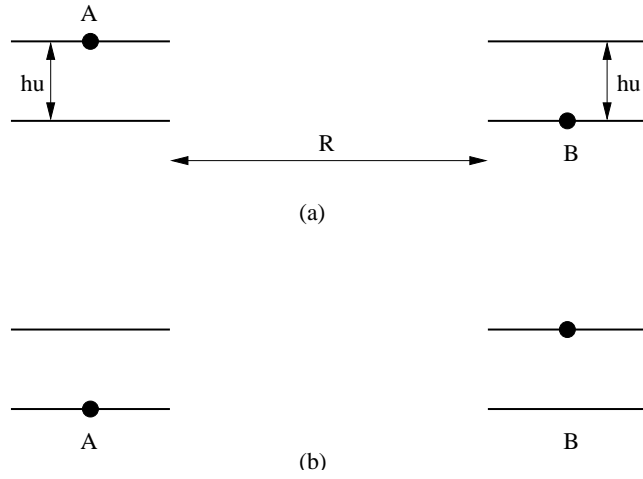


Fig. 1. Schematic representation of the (a) initial and (b) final states of Fermi's two-level system.

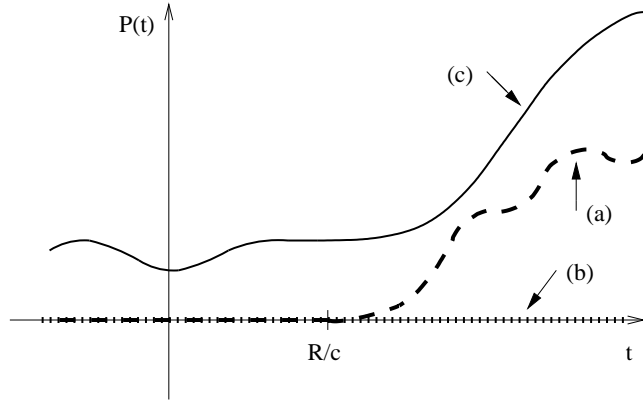


Fig. 2. Schematic representation of (a) Fermi's causal result, (b) first outcome of Hegerfeldt's theorem, and (c) second outcome of Hegerfeldt's theorem.